**Univerzitet u Zenici**
**Pedagoški fakultet**
Odsjek: Matematika i informatika
Zenica, 31.01.2013.

Pismeni ispit iz predmeta **Analiza III**

**1.** Naći ekstreme funkcije $z = (2x^2 + 3y^2)e^{-(x^2+y^2)}$ .

**2.** (a) Dati dvostruki integral $\int\limits_0^R dx \int\limits_0^{\sqrt{R^2-x^2}} f(x,y)\,dy$ iz pravougaonih koordinata transformisati na polarne koordinate.

(b) Dati trojni integral $\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz$ transformisati na trostruki u cilindričnim koordinatama (sa određenim posebnim granicama integracije) ako je $\Omega$ oblast u prvom oktantu ograničen cilindrom $x^2 + y^2 = R^2$ i ravnima $z = 0$, $z = 1$, $y = x$ i $y = x\sqrt{3}$.

**3.** Izračunati krivoliniski integral

$$I = \oint\limits_C z\,dz$$

duž krive koja nastaje kao presjek cilindra $\frac{(x-\frac{a}{2})^2}{\frac{a^2}{2}} + \frac{(y-\frac{b}{2})^2}{\frac{b^2}{2}} = 1$ i paraboloida $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ orjentisana u pozitivnom smijeru $(a \geq b > 0)$.

**4.** Izračunati površinu dijela površi $S : z^2 = 2xy$ određene u prvom oktantu u presjeku sa ravnima: $x = 0$, $y = 0$ i $x + y = 1$.
**Uputa:** $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$, $B(\frac{3}{2},\frac{3}{2}) = \frac{\pi}{8}$, $B(\frac{1}{2},\frac{5}{2}) = \frac{3\pi}{8}$.

Zadaci su skinuti sa stranice pf.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

(#) Naći ekstreme f-je $z=(2x^2+3y^2)e^{-(x^2+y^2)}$.

Rj.

$$\frac{\partial z}{\partial x} = 4x\cdot e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2x)=(4x-4x^3-6xy^2)e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y} = 6y e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2y)=(6y-4x^2y-6y^3)e^{-x^2-y^2}$$

$$2x(2-2x^2-3y^2)e^{-x^2-y^2}=0$$
$$2y(3-2x^2-3y^2)e^{-x^2-y^2}=0$$

$e^{-x^2-y^2}\neq 0 \quad \forall (x,y\in\mathbb{R})$

$x=0$ ; $y=0$, $M_1(0,0)$

ili

$x=0$ i $3-2x^2-3y^2=0$

$3y^2=3$

$y^2=1$

$y_{1,2}=\pm 1$

$M_2(0,-1)$

$M_3(0,1)$

ili

$2-2x^2-3y^2=0$ ; $y=0$

$2x^2=2$

$x^2=1$

$x_{1,2}=\pm 1$

$M_4(-1,0)$

$M_5(1,0)$

ili

$$\begin{array}{r} 2-2x^2-3y^2=0 \\ -\quad 3-2x^2-3y^2=0 \\ \hline -1=0 \end{array}$$

sistem nema rješenja

Stacionarne tačke su $M_1$, $M_2$, $M_3$, $M_4$ i $M_5$.

$$\frac{\partial^2 z}{\partial x^2}=(4-12x^2-6y^2)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2x)=(8x^4+12x^2y^2-20x^2-6y^2+4)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=(-12xy)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2y)=(-20xy+8x^3y+12xy^3)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=(6-4x^2-18y^2)e^{-x^2-y^2}+(6y-4x^2y-6y^3)e^{-x^2-y^2}\cdot(-2y)=(-30y^2+12y^4+8x^2y^2-4x^2+6)e^{-x^2-y^2}$$

za $M_1(0,0)$, $A=4$, $B=0$, $C=6$, $D=AC-B^2=24>0$ ima ekstrem

$A>0$ ima minimum, $z_{min}(0,0)=0$

za $M_2(0,-1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum, $z_{max}(0,-1)=3e^{-1}$

za $M_3(0,1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum $z_{max}(0,1)=3e^{-1}$

za $M_4(-1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$, $D=AC-B^2=-16e^{-2}<0$

f-ja u tački $M_4(-1,0)$ nema ekstrem

za $M_5(1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$,

f-ja u tački $M_5(1,0)$ nema ekstrem

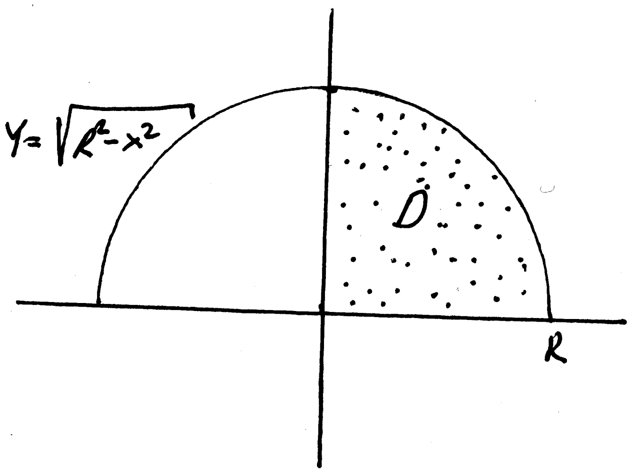
(#) Dati dvostruki integral $\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy$

iz pravouglaonih koordinata transformisati na polarne koordinate.

Rj. Oblast integracije $D$ prema postavci zadatka je

$$D: \begin{cases} 0 \le x \le R \\ 0 \le y \le \sqrt{R^2-x^2} \end{cases}$$

Skicirajmo oblast $D$.




$$y^2 = R^2 - x^2$$

$$x^2 + y^2 = R^2$$

Polarne koordinate glase

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$dx\,dy = r\,dr\,d\varphi$$

$$D \xrightarrow{\text{transformise}} D': \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy = \int_0^R r\,dr \int_0^{\pi/2} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$

(#) Dati trojni integral $\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz$

transformisati na trostruki u cilindričnim koordinatama (sa određenim posebnim granicama integracije) ako je $\Omega$ oblast u prvom oktantu ograničen cilindrom $x^2+y^2=R^2$ i ravnima $z=0$, $z=1$, $y=x$ i $y=x\sqrt{3}$

Rj.

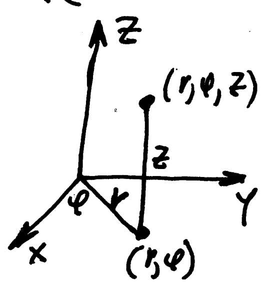
Cilindrične koordinate glase

$x = r\cos\varphi$

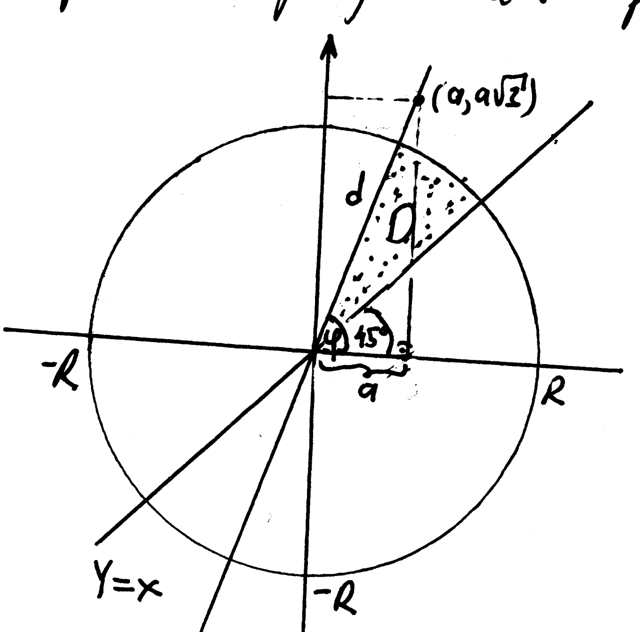
$y = r\sin\varphi$

$z = z$

$dx\,dy\,dz = r\,dr\,d\varphi\,dz$




Napravimo presjek datih površina sa $xOy$ ravni.




$\cos\varphi = \dfrac{a}{d} = \dfrac{a}{2a} = \dfrac{1}{2} \Rightarrow \varphi = \dfrac{\pi}{3}$

$d^2 = a^2 + 3a^2 = 4a^2$

$d = 2a$

Sad nije teško vidjeti da je

$$\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int_0^1 dz \int_{\pi/4}^{\pi/3} d\varphi \int_0^R f(r\cos\varphi, r\sin\varphi, z)\, r\,dr.$$

$\Omega \xrightarrow{\text{transformše}} \Omega'$

$$\Omega' : \begin{cases} 0 \le r \le R \\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{3} \\ 0 \le z \le 1 \end{cases}$$

(#) Izračunati krivolinijski integral

$$I = \oint_C z\,dz.$$

duž krive koja nastaje kao presjek cilindra $\dfrac{(x-\frac{a}{2})^2}{\frac{a^2}{2}} + \dfrac{(y-\frac{b}{2})^2}{\frac{b^2}{2}} = 1$ i paraboloida $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ orjentisana u pozitivnom smijeru $(a \geq b > 0)$.

Rj. Prisjetimo se

Ako je kriva $C: \begin{cases} x = \mu(t) \\ y = \eta(t) \\ z = \theta(t) \\ t_1 \leq t \leq t_2 \end{cases}$ data u parametarskom obliku, tada

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_{t_1}^{t_2} \big( P(\mu(t),\eta(t),\theta(t))\,\mu'(t) + Q(\mu(t),\eta(t),\theta(t))\,\eta'(t)$$
$$+ R(\mu(t),\eta(t),\theta(t))\,\theta'(t) \big)\,dt$$

Da bi izračunali dati integral trebamo parametrizirati datu krivu. Uvrstimo smjene za $x$ i $y$ t.d. $\dfrac{(x-\frac{a}{2})^2}{\frac{a^2}{2}} + \dfrac{(y-\frac{b}{2})^2}{\frac{b^2}{2}} = 1$. $(*)$

Za $x$ i $y$ mogu nam pomoći pooopštene polarne koordinate (gdje je $r$ fiksirano)

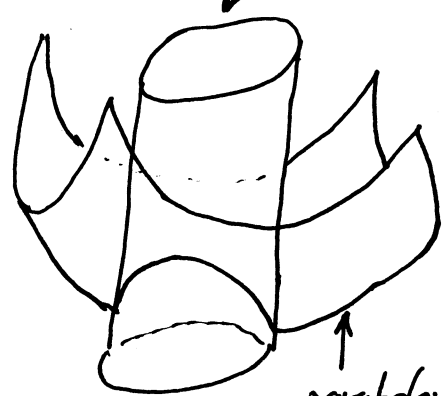
$$\left.\begin{array}{ll} x = \frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi & \Rightarrow \left(x-\frac{a}{2}\right)^2 = \frac{a^2}{2}\cos^2\varphi \\ y = \frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi & \Rightarrow \left(y-\frac{b}{2}\right)^2 = \frac{b^2}{2}\sin^2\varphi \end{array}\right\} \Rightarrow \text{vrijedi } (*) \text{ za } \varphi \in [0, 2\pi)$$

Sada je

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\left(\frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi\right)^2}{a^2} + \frac{\left(\frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi\right)^2}{b^2} = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cos\varphi\right)^2 + \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\sin\varphi\right)^2 =$$

$$= \frac{1}{4} + \frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{2}\cos^2\varphi + \frac{1}{4} + \frac{1}{\sqrt{2}}\sin\varphi + \frac{1}{2}\sin^2\varphi =$$

$$= 1 + \frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{\sqrt{2}}\sin\varphi ,$$




Prema tome imamo

$$z = 1 + \frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{\sqrt{2}}\sin\varphi$$

$$dz = -\frac{1}{\sqrt{2}}\sin\varphi + \frac{1}{\sqrt{2}}\cos\varphi$$

$$\oint_C z\,dz = \int_0^{2\pi}\left(1 + \frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{\sqrt{2}}\sin\varphi\right)\left(-\frac{1}{\sqrt{2}}\sin\varphi + \frac{1}{\sqrt{2}}\cos\varphi\right)d\varphi =$$

$$= \int_0^{2\pi}\Big(-\frac{1}{\sqrt{2}}\sin\varphi + \frac{1}{\sqrt{2}}\cos\varphi \underline{- \frac{1}{2}\sin\varphi\cos\varphi} + \underbrace{\frac{1}{2}\cos^2\varphi - \frac{1}{2}\sin^2\varphi}_{=\frac{1}{2}(\cos^2\varphi - \sin^2\varphi)} \underline{+ \frac{1}{2}\sin\varphi\cos\varphi}\Big)d\varphi$$

$$= -\frac{1}{\sqrt{2}}\int_0^{2\pi}\sin\varphi\,d\varphi + \frac{1}{\sqrt{2}}\int_0^{2\pi}\cos\varphi\,d\varphi + \frac{1}{2}\int_0^{2\pi}\cos 2\varphi\,d\varphi =$$

$$= -\frac{1}{\sqrt{2}}(-\cos\varphi)\Big|_0^{2\pi} + \frac{1}{\sqrt{2}}\sin\varphi\Big|_0^{2\pi} + \frac{1}{2}\cdot\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi} =$$

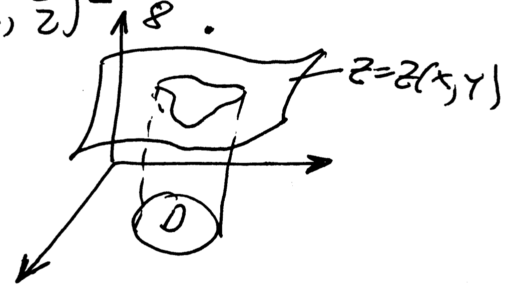
$$= \frac{1}{\sqrt{2}}(1-1) + 0 + 0 = 0$$

(#) Izračunati površinu dijela površi $S: z^2=2xy$ određene u prvom oktantu u presjeku sa ravnima: $x=0$, $y=0$ i $x+y=1$.

Uputa: $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$, $B(\frac{3}{2},\frac{3}{2})=\frac{\pi}{8}$, $B(\frac{1}{2},\frac{5}{2})=\frac{3\pi}{8}$.

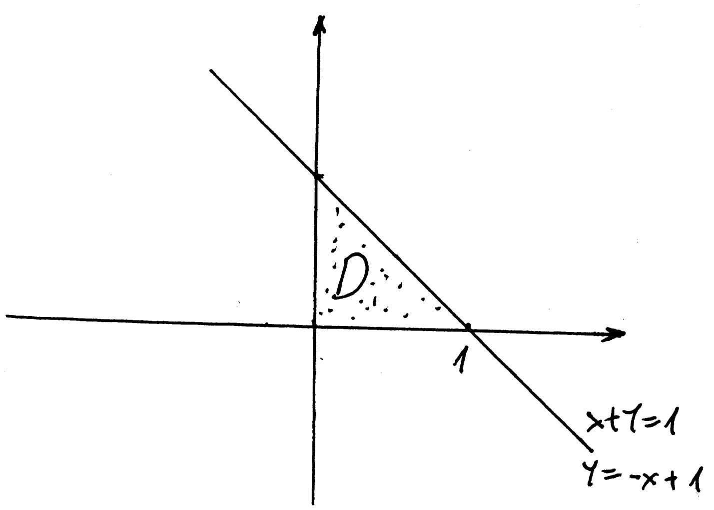
Rj:

$$P=\iint_S dS=\iint_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$




Kako je površina S u prvom oktantu, u našem slučaju je

$S:\ z=\sqrt{2}\sqrt{xy}$




$$z'_x=\sqrt{2}\,\frac{y}{2\sqrt{xy}}$$

$$z'_y=\sqrt{2}\,\frac{x}{2\sqrt{xy}}$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{y^2}{2xy}+\frac{x^2}{2xy}=\frac{2xy+y^2+x^2}{2xy}=\frac{(x+y)^2}{2xy}$$

$$P=\iint_S dS=\iint_D \frac{x+y}{\sqrt{2xy}}\,dx\,dy=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{-x+1}\frac{(x+y)}{\sqrt{xy}}\,dy=$$

$$=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{1-x}\left(x\cdot x^{-\frac{1}{2}}\cdot y^{-\frac{1}{2}}+y\cdot x^{-\frac{1}{2}}\cdot y^{-\frac{1}{2}}\right)dy=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{1-x}\left(x^{\frac{1}{2}}y^{-\frac{1}{2}}+x^{-\frac{1}{2}}y^{\frac{1}{2}}\right)dy$$

$$=\frac{1}{\sqrt{2}}\int_0^1\left(x^{\frac{1}{2}}\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^{1-x}+x^{-\frac{1}{2}}\cdot\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^{1-x}\right)dx=\frac{2}{\sqrt{2}}\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}dx+$$

$$+\frac{2}{3\sqrt{2}}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}}dx=\sqrt{2}\int_0^1 x^{\frac{3}{2}-1}(1-x)^{\frac{3}{2}}dx+\frac{\sqrt{2}}{3}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{5}{2}-1}dx=\sqrt{2}\,B\left(\frac{3}{2},\frac{3}{2}\right)+$$

$$+\frac{\sqrt{2}}{3}B\left(\frac{1}{2},\frac{5}{2}\right)=\frac{\pi}{2\sqrt{2}}.$$